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Internal wave–vortical mode interactions in strongly stratified flows

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In this paper, weakly nonlinear interactions in a strongly-stratified, inviscid flow are re-examined, taking into account the presence of both internal waves and vortical modes. We use a multiple scale formulation, based on the two characteristic times of the problem. Ertel's potential vorticity motivates a splitting of the velocity into propagating (wave) and non-propagating (vortical) contributions. We focus on the three fundamental interactions: the wave/wave, wave/vortex and vortex/vortex interactions. The oft-studied wave/wave interaction illustrates the difference between potential and vertical vorticities. We then identify two additional resonances for the wave/vortex and vortex/vortex interactions respectively. The wave/vortex resonance provides a mechanism for redistributing energy in spectral space while the vortex/vortex interaction may give rise to an internal wave field.

1. Introduction and background

In order to assess the effect of a stable stratification upon the evolution of turbulence, Lin & Pao (1979) carried out a series of experiments in which slender objects were towed in a tank which was stably stratified with salt water. Initially, the ensuing wakes behaved as in the absence of stratification. After about a quarter of a buoyancy period, however, the wakes collapsed, releasing some energy in the form of an internal wave field. From dye visualizations they observed that, after roughly five buoyancy periods, some definite coherent, mainly horizontal, meandering structures became apparent. These persisted for another five or so buoyancy periods before slowly dying away. Direct numerical simulations of the decay of stratified homogeneous turbulence by Riley, Metcalfe & Weissman (1981) have revealed similar behaviour: the stratification tends to enhance the growth of horizontal scales while suppressing vertical motion. Taking advantage of the two timescales which arise naturally in this problem, Riley *et al.* have proposed a scaling which, at low Froude number, effectively splits the velocity field into propagating (wave) and non-propagating (vortical mode) components. The formulation in terms of propagating and non-propagating motions has also been used by Staquet & Riley (1989) to analyse the later stages of decay of a strongly-stratified turbulent mixing layer. Their numerical results indicate that, for large times, the vortical mode is much more energetic than the wave mode. Staquet & Riley (1990) have also devised a numerical method to extract diagnostically the velocity field associated with a potential vorticity distribution.

At large geophysical scales, the non-propagating mode is the familiar geostrophic mode, representing a balance between the Coriolis force and the pressure gradient.

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Müller *et al.* (1986) have postulated that the observed oceanic current fine structure is a manifestation of the vortical mode at the small scales. This mode, they argue, is needed to account for the potential vorticity present in the flow (hence the name potential vorticity mode). Moreover, estimates of potential vorticity from the IWEX data reveals that the energy contained in non-wave motion is significant and contributes to the observed shear (Müller, Lien & Williams 1988). In the atmosphere, one hypothesis put forth by Gage (1979) and expanded upon by Lilly (1983), using a scale analysis similar to that of Riley *et al.* (1981), is that the observed horizontal energy spectrum at the mesoscales is due to an upscale energy transfer through stratified turbulence. As initially three-dimensional turbulence evolves, roughly half of its energy propagates away in the form of an internal wave field, while the rest is transformed to quasi-horizontal turbulence. The latter, exhibiting some of the features of two-dimensional turbulence, transfers some of its energy upscale, producing a $\kappa^{-\frac{5}{3}}$ power law energy spectrum, characteristic of two-dimensional turbulence (Kraichnan 1967). The small fraction of energy which travels to larger scales appears able to reconcile theory with observations. Gage's more recent work (Gage & Nastrom 1988) indicates the presence of a superposition of waves and vortical structures, the latter clearly apparent in the horizontal velocity fields. In a related numerical experiment, Herring & Métais (1988) have looked into the occurrence of a $\kappa^{-\frac{5}{3}}$ inverse cascade of energy for strongly-stratified three-dimensional turbulence by selectively forcing the small scales. With two-dimensional forcing, they have observed a weak inverse cascade in the horizontal as well as pronounced vertical variability. The inverse cascade and vertical variability disappear, however, as the forcing becomes three-dimensional. More recently, Dong & Yeh (1988) have examined off-resonant interactions among internal gravity waves, vortical modes and acoustic waves in an isothermal atmosphere. Using an eigenmode formulation similar to that of Müller *et al.* (1988) along with a linear stability analysis, they give an example of a strong off-resonant interaction between a primary internal wave and two secondary vortical modes. The vortical modes exhibit exponential growth and Dong & Yeh have suggested that this drain of energy from the wave may help explain the saturation of wave energy observed in the atmosphere. They have not, however, made the distinction between potential and vertical vorticities and, consequently, as we will see below, their results remain ambiguous.

The aim of this study is to assess the effect of the vortical mode on the evolution of strongly-stratified flows by examining various possible weakly nonlinear interactions. In the next section we formulate our mathematical model, using a multiple timescale formulation. Then, in the subsequent sections, we discuss weakly nonlinear wave/wave, wave/vortical mode, and vortical mode/vortical mode interactions. Our conclusions and discussion are presented in the final section.

2. Mathematical model

The governing flow equations are,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho_b}\nabla p - \frac{g}{\rho_b}\rho \boldsymbol{i}_3, \tag{1}$$

$$\frac{\mathrm{D}\tilde{\rho}}{\mathrm{D}t} = 0, \tag{2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3}$$

where $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$. Here $\boldsymbol{u}$ is the velocity vector, p is the pressure, and

$$\tilde{\rho}(x,y,z,t) = \rho_{\mathrm{b}} + \bar{\rho}(z) + \rho(\boldsymbol{x},t) \tag{4}$$

is the density, expressed as the sum of a constant reference density ρ_{b}, a linear ambient density $\bar{\rho}(z)$ and a perturbation density $\rho(\boldsymbol{x},t)$. $\boldsymbol{i}_3$ denotes the unit vector in the positive z-direction, opposite to the gravity force. We have assumed the Boussinesq approximation.

We non-dimensionalize with a single velocity scale U and single lengthscale L, both to be set for each example addressed, and one time T_0. For T_0, we choose the buoyancy period,

$$T_0 = \frac{1}{N} \tag{5}$$

where,

$$N = \left(-\frac{g}{\rho_{\mathrm{b}}}\frac{\partial\bar{\rho}}{\partial z}\right)^{\frac{1}{2}}. \tag{6}$$

The density scales as $\rho_{\mathrm{b}}NU/g$. The pressure scale is chosen in such a way that pressure gradients appear in the lowest-order equations.

We introduce Ertel's potential vorticity,

$$\Pi \equiv \boldsymbol{\zeta}\cdot\boldsymbol{\nabla}\tilde{\rho}. \tag{7}$$

In the absence of viscous and diffusive forces, Π is conserved following the motion (Ertel 1942). This means that individual fluid parcels retain their own potential vorticity. Thus, potential vorticity is a non-propagating quantity in the sense that it is not transferred between fluid particles. It can hence be used to trace the non-propagating vortical mode. We define the wave field to be precisely that part of the flow which propagates and hence does not contribute to the potential vorticity and the vortical part (non-propagating) to be accountable for all of the potential vorticity (in the absence of a mean flow). The important consequence of potential vorticity conservation for the present problem is that, if no vortical mode is present initially, none can be created through a weakly nonlinear interaction.

Physically speaking, Π gives a measure of the component of vorticity in the direction of the density gradient. When the deviations from equilibrium are slight, the constant-density (isopycnal) surfaces are, to a first approximation, horizontal. The potential vorticity is then simply proportional to the vertical vorticity. As the isopycnals steepen, however, the nonlinear contribution can no longer be neglected. Thus, only in the linear limit are vertical and potential vorticities synonymous. This correspondence between potential and vertical vorticity motivates a splitting of the velocity field (used in the present context by Riley *et al.* and by Lilly) which, in the linear limit, corresponds exactly to a wave/vortical mode decomposition.

The velocity is written in terms of a stream function $\Psi(x,y,z,t)$, a scalar velocity potential $\Phi(x,y,z,t)$ and a vertical velocity $w(x,y,z,t)$,

$$\boldsymbol{u} = \boldsymbol{i}_3 \times \boldsymbol{\nabla}_{\mathrm{h}}\Psi + \{\boldsymbol{\nabla}_{\mathrm{h}}\Phi + w\boldsymbol{i}_3\}, \tag{8}$$

where $\boldsymbol{\nabla}_{\mathrm{h}}$ is the horizontal gradient operator. The first term is horizontal and non-divergent, and accounts for all of the vertical vorticity in the flow. It is thus the linear projection of the vortical mode velocity. The second bracketed term is vertically irrotational and divergence-free, and thus represents the linear wave velocity. In the linear limit, the vortical velocity is entirely horizontal and the vertical velocity traces the wave field.

Inherent to the problem is the presence of two natural timescales, T_0 defined previously and $T_1 = L/U$, an advective timescale. The ratio of these two timescales,

$$\frac{T_0}{T_1} = \frac{(1/N)}{L/U} = \frac{U}{NL}, \tag{9}$$

defines a Froude number. When the ambient density gradient is large, N is large also. In this case, two distinct timescales emerge, and the Froude number appears as a small parameter ϵ.

Associated with T_0 and T_1 are dimensionless times,

$$t_0 = t/T_0 \quad \text{a 'fast' time,} \tag{10}$$

and,

$$t_1 = t/T_1 = \epsilon t_0 \quad \text{a 'slow' time.} \tag{11}$$

The crux of the multiple scale method consists of treating t_0 and t_1 as independent variables. Then the time derivative is rewritten,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1}. \tag{12}$$

We expand all flow variables in power series of ϵ. The uniform validity of the solutions for sufficiently large times is ensured by imposing the constraint that each of the coefficients in the power series remain of $O(1)$ for all times (see, e.g. Kevorkian & Cole 1981). Upon substitution of (8) into (1)–(3) and after some algebraic manipulations, we obtain at the lowest order,

$$L(w_0) \equiv \frac{\partial^2}{\partial t_0^2} \nabla^2 w_0 + \nabla_h^2 w_0 = 0, \tag{13}$$

$$M(\Psi_0) \equiv \frac{\partial}{\partial t_0} \nabla_h^2 \Psi_0 = 0, \tag{14}$$

$$N(b_0) \equiv \frac{\partial b_0}{\partial t_0} + w_0 = 0. \tag{15}$$

Equation (13) is the familiar linear internal wave equation, (14) is the linear vertical vorticity equation and (15) is the buoyancy equation. At this order, wave and vortical fields are decoupled and the slow timescale does not appear explicitly. Also notice that the vertical vorticity remains constant on the fast timescale. Since, to lowest order, potential and vertical vorticities are proportional, this is consistent with the requirement that the potential vorticity be conserved at each order.

At the next order of approximation, we have

$$L(w_1) = -2 \frac{\partial^2}{\partial t_0 t_1} \nabla^2 w_0 + \mathscr{I}_{ww} + \mathscr{I}_{wv} + \mathscr{I}_{vv}, \tag{16}$$

$$M(\psi_1) = -\frac{\partial(\Psi_0, \nabla_h^2 \Psi_0)}{\partial(x,y)} - \frac{\partial}{\partial t_1} \nabla_h^2 \Psi_0 + \mathscr{J}_{ww} + \mathscr{J}_{wv}, \tag{17}$$

$$N(b_1) = -\frac{\partial b_0}{\partial t_1} + \mathscr{K}_{ww} + \mathscr{K}_{wv}, \tag{18}$$

with the nonlinear wave/wave, wave/vortical and vortical/vortical coupling terms appearing on the right-hand sides.

The linear problem has three independent eigenmodes. Two are associated with internal waves with corresponding eigenfrequencies $\omega_{\pm} = \pm\kappa_h/(\kappa_h^2+m^2)^{\frac{1}{2}}$. Here $\boldsymbol{\kappa}$ is a wavenumber vector with horizontal and vertical components $\boldsymbol{\kappa}_h = (k, l)$ and m, respectively, and $\kappa_h = |\boldsymbol{\kappa}_h|$. The third has eigenfrequency $\omega_0 = 0$; it represents the non-propagating vortical mode. Initial conditions for the three cases addressed in this paper will consist of linear combinations of these three eigenmodes.

3. The wave/wave interaction

The purpose of this first case is to illustrate how a wave/wave interaction can lead to a production of vertical vorticity without violating the conservation of potential vorticity. Hence no vortical mode component is generated by this interaction. The initial configuration is made up of two waves with wavevectors $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2$ and amplitudes A_1, A_2 and no vortical modes.

$$w(\boldsymbol{x},0,0) = A_1 \sin(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}) + A_2 \sin(\boldsymbol{\kappa}_2\cdot\boldsymbol{x}), \tag{19}$$

$$\frac{\partial w}{\partial t}(\boldsymbol{x},t_0,t_1)\big|_{t=0} = -A_1\omega_1 \cos(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}) - A_2 w_2 \cos(\boldsymbol{\kappa}_2\cdot\boldsymbol{x}), \tag{20}$$

$$\nabla_h^2\,\Psi(\boldsymbol{x},0,0) = 0, \tag{21}$$

$$b(\boldsymbol{x},0,0) = -\frac{A_1}{\omega_1}\cos(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}) - \frac{A_2}{\omega_2}\cos(\boldsymbol{\kappa}_2\cdot\boldsymbol{x}), \tag{22}$$

where ω_1 and ω_2 are the respective internal wave frequencies of modes κ_1 and κ_2. Pressure is scaled in such a way that the pressure gradient terms are retained at the lowest order. This amounts to picking $[P] = \rho_b NUL$. From (14), we can only conclude that Ψ_0 is independent of t_0,

$$\Psi_0(\boldsymbol{x},t_0,t_1) = \Psi_0(\boldsymbol{x},t_1), \tag{23}$$

with,

$$\Psi_0(\boldsymbol{x},0) = 0. \tag{24}$$

For the vertical velocity, we assume a simple solution of the form,

$$w_0(\boldsymbol{x},t_0,t_1) = \alpha_1(t_1)\sin\theta_1 + \alpha_2(t_1)\sin\theta_2 \tag{25}$$

where the phase function θ is defined as,

$$\theta_i \equiv \boldsymbol{\kappa}_i\cdot\boldsymbol{x} - \omega_i t_0 + \xi_i(t_1) \quad (i = 1, 2).$$

The amplitudes α_1, α_2 and the phases ξ_1, ξ_2 are real functions of the slow time t_1. Imposition of the initial conditions yields, $\alpha_1(0) = A_1$, $\alpha_2(0) = A_2$ and $\xi_1(0) = 0$, $\xi_2(0) = 0$. From the continuity relation, we obtain an expression for Φ_0 and solving (15) gives b_0. The calculation must be carried out to the next order to get the t_1 dependence of the solutions.

The $O(\epsilon)$ initial conditions are,

$$w_1(\boldsymbol{x},0,0) = 0, \tag{26}$$

$$\Psi_1(\boldsymbol{x},0,0) = 0, \tag{27}$$

$$b_1(\boldsymbol{x},0,0) = 0, \tag{28}$$

$$\left.\frac{\partial w_1}{\partial t_0}\right|_{t=0} = -\left.\frac{\partial w_0}{\partial t_1}\right|_{t=0}. \tag{29}$$

We proceed by first considering (17). Since Ψ_0 is not a function of t_0, the first two terms on the right-hand side are independent of t_0 and thus satisfy the homogeneous equation. If retained, they will give rise to a particular solution Ψ_p which will be linear in t_0. As t_0 becomes $O(1/\epsilon)$, $\epsilon\Psi_p$ will become $O(1)$ and the asymptotic series for Ψ will no longer be uniformly valid. In order to avoid this breakdown of our solution, we must eliminate such secular terms from the right-hand side. This is achieved by setting

$$\frac{\partial}{\partial t_1}\nabla_h^2 \Psi_0+\frac{\partial(\Psi_0,\nabla_h^2\Psi_0)}{\partial(x,y)}=0. \tag{30}$$

We pause for a moment and see what can be deduced from our present knowledge of Ψ_0. We know that,

$$\Psi(\boldsymbol{x},0)\,|_{t=0}=0. \tag{31}$$

In addition,

$$\frac{\partial \Psi_0}{\partial t_0}\equiv 0, \tag{32}$$

and (30) tells us that Ψ_0 is conserved following the motion on the slow timescale. This implies that

$$\Psi_0(\boldsymbol{x},t_1)\equiv 0. \tag{33}$$

Thus, the only forcing term in (17) is $\mathscr{J}_{ww}$. Upon substitution of the lowest-order solutions into the right-hand side of (17), we obtain

$$M(\psi_1)=\tfrac{1}{2}\alpha_1\alpha_2\left(\frac{m_1^2}{\kappa_{h_1}^2}-\frac{m_2^2}{\kappa_{h_2}^2}\right)(k_1 l_2-k_2 l_1)\{\cos\theta_{12}^{+}+\cos\theta_{12}^{-}\}, \tag{34}$$

where,

$$\theta_{12}^{\pm}=(\boldsymbol{\kappa}_1\pm\boldsymbol{\kappa}_2)\cdot\boldsymbol{x}-(\omega_1\pm\omega_2)t_0+(\xi_1\pm\xi_2). \tag{35}$$

Here $\kappa_i=(k_i,l_i,m_i)$, with $i=1,2$. Physically, the remaining terms on the right-hand side represent forcing of the vertical vorticity due to the vortex turning mechanism, the velocity field of each wave turning the other wave's horizontal vorticity toward the vertical. The solution is readily written down,

$$\nabla_h^2\Psi_1=\tfrac{1}{2}\alpha_1\alpha_2\left(\frac{m_1^2}{\kappa_{h_1}^2}-\frac{m_2^2}{\kappa_{h_2}^2}\right)(k_1 l_2-k_2 l_1)\left\{\frac{1}{\omega_1+\omega_2}\sin\theta_{12}^{+}+\frac{1}{\omega_1-\omega_2}\sin\theta_{12}^{-}\right\}+C_2(\boldsymbol{x},t_1). \tag{36}$$

The first two terms represent the particular solution and C_2 is the homogeneous solution. Imposition of the initial condition yields,

$$C_2(\boldsymbol{x},0)=\frac{1}{2}\left(\frac{m_1^2}{\kappa_{h_1}^2}-\frac{m_2^2}{\kappa_{h_2}^2}\right)(k_1 l_2-k_2 l_1)\times\left\{\frac{1}{\omega_1+\omega_2}\sin\left[(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\boldsymbol{x}\right]+\frac{1}{\omega_1-\omega_2}\sin\left[(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2)\cdot\boldsymbol{x}\right]\right\}. \tag{37}$$

The expression for $\nabla_h^2\Psi_1$ vanishes if either

$$k_1 l_2=k_2 l_1, \tag{38}$$

or,

$$\frac{m_1^2}{\kappa_{h_1}^2}=\frac{m_2^2}{\kappa_{h_2}^2}. \tag{39}$$

If (38) is satisfied, the two wave vectors lie in the same vertical plane. Consequently, their associated internal wave velocities also lie in this plane while the vorticities are horizontal and out of the plane. In the out-of-plane direction, the velocity vectors are constant and the term responsible for the vorticity turning identically vanishes,

$$\boldsymbol{\zeta}_{\mathrm{h}}\cdot\boldsymbol{\nabla}w \equiv 0. \tag{40}$$

Thus, no turning results.

If (39) is satisfied, the two waves have equal frequencies as can be seen by rewriting (39) as,

$$\frac{m_1^2}{\kappa_{\mathrm{h}_1}^2}+\frac{\kappa_{\mathrm{h}_1}^2}{\kappa_{\mathrm{h}_1}^2}=\frac{m_2^2}{\kappa_{\mathrm{h}_2}^2}+\frac{\kappa_{\mathrm{h}_2}^2}{\kappa_{\mathrm{h}_2}^2}, \tag{41}$$

which is equivalent to,

$$\frac{1}{\omega_1^2}=\frac{1}{\omega_2^2}. \tag{42}$$

In this case, the turning of the horizontal vorticity of wave 1 by the vertical velocity of wave 2 is exactly equal and opposite to the turning of the horizontal vorticity of wave 2 by the vertical velocity of wave 1. Thus, no net vertical vorticity is produced. We note that the expression for $\nabla_{\mathrm{h}}^2 \Psi_1$ remains well-behaved since the vanishing of one of its denominators $(\omega_1 \pm \omega_2 = 0)$ is always accompanied by the vanishing of the numerator. Hence, for all wave vectors $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, Ψ_1 remains bounded.

Moreover, note that the presence of a non-zero $O(\epsilon)$ correction to the vertical vorticity does not violate Ertel's theorem. Consider the scaled version of the potential vorticity equation,

$$\frac{\partial \zeta_{0_z}}{\partial t_0}+\epsilon\left\{\frac{\partial \zeta_{1_z}}{\partial t_0}+\frac{\partial \zeta_{0_z}}{\partial t_1}+\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\zeta_{0_z}+\frac{\partial}{\partial t_0}(\boldsymbol{\zeta}_0\cdot\boldsymbol{\nabla}b_0)\right\}+O(\epsilon^2)=0. \tag{43}$$

In order for the multiple scale formalism to be consistent, potential vorticity must be conserved at each order.

At the lowest order, this constrains the vertical vorticity ζ_{0_z} to be independent of t_0. At $O(\epsilon)$, the bracketed expression must vanish, or

$$\frac{\partial \zeta_{1_z}}{\partial t_0}=-\frac{\partial \zeta_{0_z}}{\partial t_1}-\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\zeta_{0_z}-\frac{\partial}{\partial t_0}(\boldsymbol{\zeta}_0\cdot\boldsymbol{\nabla}b_0). \tag{44}$$

It is straightforward to show, by manipulating the equations, that (44) is exactly (17). In this particular example, some vertical vorticity is produced by the interaction of two waves yet the potential vorticity, and hence the vortical field, remains zero. The difference between potential and vertical vorticities has, we feel, been overlooked by Dong & Yeh. In their formulation, the non-zero vertical vorticity would be interpreted as a manifestation of vortical mode activity.

We return to (16) and (18). Upon substitution of the lowest-order solutions, they become

$$L(w_1)=\sum_{i=1}^{2}-2\omega_i\kappa_i^2\left\{\frac{\partial \alpha_i}{\partial t_1}\cos\theta_i-\alpha_i\frac{\partial \xi}{\partial t_1}\sin\theta_i\right\}+\alpha_1\alpha_2\{\Gamma_{12}^{+}\cos\theta_{12}^{+}+\Gamma_{12}^{-}\cos\theta_{12}^{-}\} \tag{45}$$

$$N(b_1)=\sum_{i=1}^{2}\frac{1}{\omega_i}\left\{\alpha_i\frac{\partial \xi_i}{\partial t_1}\sin\theta_i-\frac{\partial \alpha_i}{\partial t_1}\cos\theta_i\right\}+\alpha_1\alpha_2\{\Omega_{12}^{+}\cos\theta_{12}^{+}+\Omega_{12}^{-}\cos\theta_{12}^{-}\}, \tag{46}$$

where expressions for the terms in Γ and Ω are,

$$\Gamma_{12}^{\pm} = \pm\frac{1}{2}\left(\frac{m_2}{\kappa_{h_2}^2}(k_1 k_2 + l_1 l_2) - m_1\right)\left[\frac{m_1}{\kappa_{h_1}^2}(\kappa_{h_1}^2 \pm (k_1 k_2 + l_1 l_2))(m_1 \pm m_2)(\omega_1 \pm \omega_2) + \|\kappa_1 + \kappa_2\|_h^2(\omega_1 \pm \omega_2) + \frac{1}{2\omega_1}\|\kappa_1 + \kappa_2\|_h^2\right], \quad (47)$$

and,

$$\Omega_{12}^{\pm} = \pm\left(\frac{m_1}{2\omega_1} - \frac{m_2}{2\kappa_{h_2}^2}(k_1 k_2 + l_1 l_2)\right). \quad (48)$$

The first four terms on the right-hand side are solutions of the homogeneous operator. If retained, they will lead to unbounded growth of the solution and hence must be discarded.

The fifth term satisfies the homogeneous equation if

$$\omega(\kappa_1 + \kappa_2) = \omega(\kappa_1) + \omega(\kappa_2). \quad (49)$$

This is the condition which the frequencies must satisfy for a resonant interaction. Similarly, the sixth term is secular if

$$\omega(\kappa_1) - \omega(\kappa_2) = \omega(\kappa_1 - \kappa_2). \quad (50)$$

If neither resonance condition holds, the elimination of secular terms leads to the conclusion that α_1, α_2, ξ_1 and ξ_2 all remain constant on the slow timescale. In this case, the interaction has at most an $O(\epsilon)$ effect on the initial waves. If, on the other hand, one of the resonance conditions, e.g. (49), is satisfied, then a third wave with wave vector $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2$ will be excited and it will periodically exchange energy with waves $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ on the slow timescale. In this case, we must assume a different form of solution for w_0, one which will allow the excited wave to become as large as the two primary waves. The elimination of secular terms then leads to coupled evolution equations for the three wave amplitudes. We shall not, at this point, include the details of the resonance calculation. The purpose of this example was not to rederive previously established results but rather to illustrate the importance of distinguishing between potential and vertical vorticities in interaction problems. The interested reader should refer to Phillips (1981) or Bretherton (1964) for a thorough treatment of wave/wave resonances.

4. The wave/vortical mode interaction

In this section, we consider the interaction of one internal wave and one vortical mode. Of particular interest is whether a strong interaction, analogous to the wave/wave resonant interaction, can be found.

We proceed in the usual fashion by imposing initial conditions on w, $\partial w/\partial t$ and Ψ. For the vortical field,

$$\Psi(\boldsymbol{x}, 0, 0) = B \sin(\boldsymbol{\kappa}_2 \cdot \boldsymbol{x}). \quad (51)$$

For the wave field,

$$w(\boldsymbol{x}, 0, 0) = A \sin(\boldsymbol{\kappa}_1 \cdot \boldsymbol{x}), \quad (52)$$

$$\frac{\partial w}{\partial t}(\boldsymbol{x}, t_0, t_1)\,|t=0 = -A\omega_1 \cos(\boldsymbol{\kappa}_1 \cdot \boldsymbol{x}), \quad (53)$$

and for the buoyancy,

$$b(\boldsymbol{x}, 0, 0) = -\frac{A}{\omega_1}\cos(\boldsymbol{\kappa}_1 \cdot \boldsymbol{x}). \quad (54)$$

The pressure is scaled, as in the last section, in such a way that the pressure gradient terms appear in the lowest-order equations.

The $O(1)$ initial conditions are,

$$w_0(\boldsymbol{x},0,0) = \sin(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}), \tag{55}$$

$$\frac{\partial w_0}{\partial t_0}(\boldsymbol{x},t_0,t_1)\,|t=0 = -\omega_1\cos(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}), \tag{56}$$

$$\nabla_{\mathrm{h}}^2\,\Psi_0(\boldsymbol{x},0,0) = -B\kappa_{\mathrm{h}_2}^2\sin(\boldsymbol{\kappa}_2\cdot\boldsymbol{x}), \tag{57}$$

$$b_0(\boldsymbol{x},0,0) = -\frac{1}{\omega_1}\cos(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}). \tag{58}$$

The vertical vorticity is readily written down,

$$\nabla_{\mathrm{h}}^2\,\Psi_0(\boldsymbol{x},t_1) = -\kappa_{\mathrm{h}_2}^2\beta(t_1)\sin\theta_2, \tag{59}$$

with,

$$\theta_2 = \boldsymbol{\kappa}_2\cdot\boldsymbol{x}+\mu(t_1), \tag{60}$$

and, from the initial conditions $\beta(0) = B$ and $\mu(0) = 0$.

For the wave field, we have

$$w_0(\boldsymbol{x},t_0,t_1) = \alpha(t_1)\sin\theta_1, \tag{61}$$

where,

$$\theta_1 = \boldsymbol{\kappa}_1\cdot\boldsymbol{x}-\omega_1 t_0+\xi(t_1), \tag{62}$$

and, from the initial data, $\alpha(0) = 1$ and $\xi(0) = 0$.

The velocity potential, Φ_0, is readily found from the continuity relation,

$$\Phi_0(\boldsymbol{x},t_0,t_1) = \frac{m_1\alpha(t_1)}{\omega_1}\cos\theta_1. \tag{63}$$

For the buoyancy, we have

$$b_0(\boldsymbol{x},t_0,t_1) = -\frac{1}{\omega_1}\cos\theta_1+C_1(\boldsymbol{x},t_1), \tag{64}$$

with,

$$C_1(\boldsymbol{x},0) = 0. \tag{65}$$

As in the previous section, the t_1 dependence of the amplitudes and phases is obtained by carrying out the calculation to $O(\epsilon)$. Owing to the transverse nature of the eigenmode velocities ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$), the self-advection of a single wave or a single vortical mode is identically zero. Therefore, the only non-zero forcing at $O(\epsilon)$ is due to the wave/vortical mode interaction. Upon substitution of the expressions for the lowest-order solutions into (16) and the regrouping of like terms, we have,

$$L(w_1) = -2\kappa_1^2\omega_1\left\{\frac{\partial\alpha}{\partial t_1}\cos\theta_1-\alpha\frac{\partial\xi}{\partial t_1}\sin\theta_1\right\}+\alpha\beta\Gamma_{12}^{+}\sin\theta_{12}^{+}+\alpha\beta\Gamma_{21}^{-}\sin\theta_{12}^{-}, \tag{66}$$

where,

$$\theta_{12}^{\pm} = (\boldsymbol{\kappa}_1\pm\boldsymbol{\kappa}_2)\cdot\boldsymbol{x}-\omega(\kappa_1)\,t_0+(\xi\pm\mu), \tag{67}$$

and,

$$\Gamma_{12}^{\pm} = \tfrac{1}{2}(k_1 l_2-k_2 l_1)\left\{\left(m_2\pm m_1\right)\omega\left(\frac{2m_1(k_1k_2+l_1l_2)}{\kappa_{\mathrm{h}_1}^2}-(m_2\mp m_1)\right)+\|\kappa_1\pm\kappa_2\|_{\mathrm{h}}^2\left(\omega+\frac{1}{\omega}\right)\right\}. \tag{68}$$

We proceed to eliminate secular terms. The first two terms on the right-hand side of (66) always satisfy the homogeneous operator and will need to be discarded. The third term on the right-hand side of (66) is a solution of the homogeneous operator only if

$$\omega(\kappa_1) = \omega(\kappa_1+\kappa_2). \tag{69}$$

Similarly, the fourth term must be discarded if,

$$\omega(\kappa_1) = \omega(\kappa_1-\kappa_2). \tag{70}$$

These are the resonance conditions for the wave/vortical mode interaction. If neither condition is satisfied, the elimination of the secular terms implies that,

$$\frac{\partial \alpha}{\partial t_1} = 0, \tag{71}$$

$$\frac{\partial \xi}{\partial t_1} = 0. \tag{72}$$

Thus, both the amplitude and the phase of the initial wave remain constant on the slow timescale when there is no resonance. The expression for w_1 is readily written down:

$$\begin{aligned} w_1 = &\frac{f_1(\kappa_1,\kappa_2)}{\omega_1^2\|\kappa_1+\kappa_2\|^2-\|\kappa_{1_h}+\kappa_{2_h}\|^2}\sin\theta_{12}^+ \\ &+\frac{f_2(\kappa_1,\kappa_2)}{\omega_1^2\|\kappa_1-\kappa_2\|^2-\|\kappa_{1_h}-\kappa_{2_h}\|^2}\sin\theta_{12}^- \\ &+\sum_n \gamma_n^+ \sin(\boldsymbol{\kappa}_n\cdot\boldsymbol{x}-\omega(\kappa_n)t_0+\xi_n^+(t_1)) \\ &+\sum_n \gamma_n^- \sin(\boldsymbol{\kappa}_n\cdot\boldsymbol{x}+\omega(\kappa_n)t_0+\xi_n^-(t_1)). \end{aligned} \tag{73}$$

The first two terms comprise the particular solution while the last two form the homogeneous solution. This expression remains uniformly valid for times of $O(1/\epsilon)$ as long as the denominators of the particular solution are not of $O(\epsilon)$, i.e. as long as $\omega_1^2\|\kappa_1+\kappa_2\|^2-\|\kappa_{1_h}+\kappa_{2_h}\|^2$ is not $O(\epsilon)$. The vanishing of one of the denominators signals a resonance. The coefficients f_1 and f_2 are of similar form as the $\Gamma_2^{\pm}$ of the previous section and will not be repeated here.

As in the wave/wave resonance of last section, we have a region of width $O(\epsilon)$ about the exact resonance within which our solution breaks down. As in the last section, we first finish the treatment of the non-resonant case before approaching the resonant one.

We now substitute the lowest-order solutions into the vertical vorticity equation,

$$M(\Psi_1) = \kappa_{h_2}^2\left\{\frac{\partial \beta}{\partial t_1}\sin\theta_2+\beta\frac{\partial \mu}{\partial t_1}\cos\theta_2\right\}+A_{12}^+\alpha_1\beta_2\sin\theta_{12}^+ + A_{12}^-\alpha_1\beta_2\sin\theta_{12}^-, \tag{74}$$

with the wave/vortex interaction coefficients,

$$A_{12}^{\pm} = \frac{1}{2}\left\{(k_1k_2+l_1l_2)\left(\frac{m_1\kappa_{h_2}^2}{\kappa_{h_1}^2}\mp m_2\right)+\kappa_{h_2}^2(m_2\pm m_1)\right\}. \tag{75}$$

All right-hand side terms which are independent of t_0 are secular and must be set equal to zero in order to preserve consistency of the asymptotic series. In the non-resonant case, only the first two right-hand terms are secular. Hence, amplitude and phase of the vertical mode remain constant on the slow timescale as well.

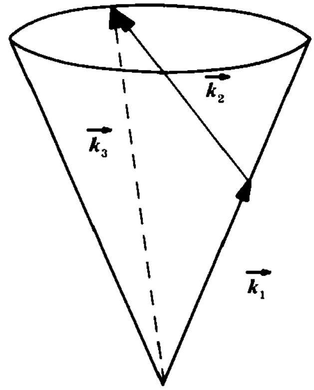

FIGURE 1. Wave/vortex resonant triad.

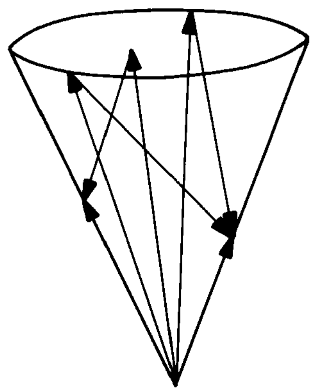

FIGURE 2. Examples of wave/vortex resonant triads.

The $O(\epsilon)$ correction to the vertical vorticity is:

$$\nabla_h^2 \Psi_1 = \frac{A_{12}^+}{\omega_1} B \sin \theta_{12}^+ + \frac{A_{12}^-}{\omega_1} B \sin \theta_{12}^- + g(\boldsymbol{x}.\, t_1). \tag{76}$$

This expression is well-behaved everywhere. If ω_1 is zero, the numerator vanishes as well so there is no problem there. $g(\boldsymbol{x}, t_1)$ is the homogeneous solution. Its temporal dependence can be determined by carrying out the analysis to $O(\epsilon^2)$ but this will not be done here.

The elimination of the secular term from the buoyancy equation requires, as in the preceding section, that the integration constant C_1 is only a function of the spatial variables. We do not pursue this any further. The interesting case is clearly the resonant one. The geometrical significance of an $O(\epsilon)$ denominator in (73) is most easily illustrated with a diagram of the resonant triad against the backdrop of a constant-frequency surface in wavenumber space, as shown in figure 1. The frequency of linear internal waves is independent of the wavenumber magnitude and depends only on the angle that the wavenumber makes with the vertical axis. Therefore, these surfaces are circular vertical cones and the condition $\omega(\kappa_1) = \omega(\kappa_1 + \kappa_2)$ implies that $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \equiv \boldsymbol{\kappa}_3$ lie on the same cone.

The vortical mode κ_2 along with the two waves κ_1 and κ_3 form a resonant triad. As demonstrated in figure 2, there is an infinite number of possible resonant triads. We proceed with a method analogous to the one used for wave/wave resonant interactions and assume a form for the lowest-order solutions that will permit the κ_3 wave to become of $O(1)$. The form of w_0 is now taken as,

$$w_0 = \alpha_1(t_1)\sin\theta_1 + \alpha_3(t_1)\cos\theta_3, \tag{77}$$

where,

$$\theta_3 = \boldsymbol{\kappa}_3\cdot\boldsymbol{x} - \omega_1 t_0 + \xi_3. \tag{78}$$

Note that we have modified our notation slightly (α is now α_1 and ξ is ξ_1). Since the two waves have equal frequencies, $\omega(\kappa_1) = \omega(\kappa_3)$, and we use ω_1 to denote the frequency of either wave. The $\frac{1}{2}\pi$ shift between the first and third waves was added for convenience.

In accordance with the initial conditions,

$$\alpha_1(0) = 1, \tag{79}$$

$$\xi_1(0) = 0, \tag{80}$$

$$\alpha_3(0) = 0, \tag{81}$$

$$\xi_3(0) = 0. \tag{82}$$

With the new lowest-order solution, the $O(\epsilon)$ wave equation is,

$$\begin{aligned} L(w_1) = & -2\kappa_1^2\omega_1\left\{\frac{\partial\alpha_1}{\partial t_1}\cos\theta_1 - \alpha_1\frac{\partial\xi_1}{\partial t_1}\sin\theta_1\right\} + 2\kappa_3^2\omega_1\left\{\frac{\partial\alpha_3}{\partial t_1}\sin\theta_3 + \alpha_3\frac{\partial\xi_3}{\partial t_1}\cos\theta_3\right\} \\ & + \Gamma_{12}^{+}\alpha_1\beta_2\sin((\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\boldsymbol{x} - \omega_1 t_0 + (\xi_1+\mu_2)) \\ & + \Gamma_{12}^{-}\sin((\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2)\cdot\boldsymbol{x} - \omega_1 t_0 + (\xi_1-\mu_2)) \\ & + \Gamma_{23}^{+}\alpha_3\beta_2\cos((\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_2)\cdot\boldsymbol{x} - \omega_1 t_0 + (\xi_3-\mu_2)) \\ & + \Gamma_{23}^{-}\cos(\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_2)\cdot\boldsymbol{x} - \omega_1 t_0 + (\xi_3+\mu_2)) \\ & + \Gamma_{13}^{+}\alpha_1\alpha_3\cos\theta_{13}^{+} + \Gamma_{13}^{-}\cos\theta_{13}^{-}. \end{aligned} \tag{83}$$

In addition to wave/vortex forcing, we also have wave/wave terms, representing the interaction of the two waves. These are of the same form as the forcing terms examined in the last section. Their effect will remain of $O(\epsilon)$ unless a third wave exists with which κ_1 and κ_3 can form a resonant triad,

$$\omega_1 + \omega_3 = \omega(\kappa_1 + \kappa_3). \tag{84}$$

We shall not consider this possibility, as the analysis then reverts back to the one carried out in the previous section. To eliminate secular terms, we define some complex amplitudes,

$$\alpha_i\cos\xi_i \equiv \mathrm{Re}\,(a_i) \equiv a_{i_\mathrm{R}} \quad \text{for} \quad i = 1,3, \tag{85}$$

$$\alpha_i\sin\xi_i \equiv \mathrm{Im}\,(a_i) \equiv a_{i_\mathrm{I}} \quad \text{for} \quad i = 1,3, \tag{86}$$

and,

$$\beta_2\cos\mu_2 \equiv \mathrm{Re}\,(b_2) \equiv b_{2_\mathrm{R}}, \tag{87}$$

$$\beta_2\sin\mu_2 \equiv \mathrm{Im}\,(b_2) \equiv b_{2_\mathrm{I}}. \tag{88}$$

We also define,

$$\theta_i = \Theta_i + \eta(t_1), \tag{89}$$

where $\eta(t_1)$ represents the combination of slowly-varying phases.

Now

$$(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)\cdot\boldsymbol{x} - \omega_1 t_0 = \boldsymbol{\kappa}_3\cdot\boldsymbol{x} - \omega_1 t_0, \tag{90}$$

and, $$(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_2)\cdot\boldsymbol{x}-\omega_1 t_0 = \boldsymbol{\kappa}_1\cdot\boldsymbol{x}-\omega_1 t_0. \tag{91}$$

The elimination of secular terms is carried out by setting the coefficients of $\cos\Theta_i$ and $\sin\Theta_i$ ($i = 1,3$) respectively equal to zero. Combining the equations for the real and imaginary parts of a_1 into a single equation, we obtain

$$2\kappa_1^2\omega_1\frac{\partial a_1}{\partial t_1} = \Gamma_{23}^- a_3 b_2^*. \tag{92}$$

An equation for a_3 is obtained in the same fashion,

$$2\kappa_3^2\omega_1\frac{\partial a_3}{\partial t_1} = -\Gamma_{12}^- a_1 b_2 \tag{93}$$

with the Γ terms written down in a symmetric fashion as functions of κ_1 and κ_3,

$$\Gamma_{23}^- = \tfrac{1}{2}(k_3 l_1 - k_1 l_3)\left\{\frac{m_1 m_3\omega_1(2k_1k_3+2l_1l_3)}{\kappa_{h_3}^2}+\omega_1(\kappa_{h_1}^2-m_1^2)+\frac{\kappa_{h_1}^2}{\omega_1}\right\}, \tag{94}$$

$$\Gamma_{21}^+ = \tfrac{1}{2}(k_3 l_1 - k_1 l_3)\left\{\frac{m_1 m_3\omega_1(2k_1k_3+2l_1l_3)}{\kappa_{h_1}^2}+\omega_1(\kappa_{h_3}^2-m_3^2)+\frac{\kappa_{h_3}^2}{\omega_1}\right\}. \tag{95}$$

By noting the following simple relationship,

$$\frac{\Gamma_{23}^-}{2\kappa_1^2\omega_1} = \frac{\Gamma_{12}^+}{2\kappa_3^2\omega_1}, \tag{96}$$

and defining

$$\Gamma' \equiv -\frac{\Gamma_{23}^-}{2\kappa_1^2\omega_1} = \tfrac{1}{2}(k_3 l_3 - k_3 l_1)\left\{\frac{m_1 m_3}{\kappa_{h_3}^2\kappa_1^2}(k_1k_3+l_1l_3)+\omega_1^2\right\}, \tag{97}$$

equations (92) and (93) can be simplified to,

$$\frac{\partial a_1}{\partial t_1} = -\Gamma' a_3 b_2, \tag{98}$$

$$\frac{\partial a_3}{\partial t_1} = \Gamma' a_1 b_2^*. \tag{99}$$

This set of equations for the two wave amplitudes is also coupled to the vortical mode amplitude. The slow time behaviour of the vortical mode is inferred from the $O(\epsilon)$ vertical vorticity equation,

$$\begin{aligned} M(\Psi_1) = \kappa_{h_2}^2&\left\{\frac{\partial b_{2R}}{\partial t_1}\sin\Theta_2+\frac{\partial b_{2I}}{\partial t_1}\cos\Theta_2\right\}+\Lambda_{13}\alpha_1\alpha_3\{\sin(\theta_1+\theta_3)\\ &-(\sin\Theta_2\cos(\xi_3-\xi_1)+\cos\Theta_2\sin(\xi_3-\xi_1))\}\\ &+\Lambda_{12}\alpha_1\beta_2\sin(\theta_1+\theta_2)+\Lambda_{23}\alpha_3\beta_2\sin(\theta_1-\theta_2), \end{aligned} \tag{100}$$

where only the secular terms have been expressed in terms of b_{2_R} and b_{2_I}. The elimination of secular terms yields an evolution equation for b_2,

$$\frac{\partial b_2}{\partial t_1} = \Lambda_{13} a_1^* a_3, \tag{101}$$

where
$$\Lambda_{13} = \tfrac{1}{2}(k_3 l_1 - k_1 l_3)\left(\frac{m_1^2}{\kappa_{h_1}^2} - \frac{m_3^2}{\kappa_{h_3}^2}\right). \tag{102}$$

Λ_{13} vanishes when $\omega_1 = \omega_3$, as can be seen by rewriting the second factor in (102) as,

$$\left\{\frac{m_1^2}{\kappa_{h_1}^2} + \frac{\kappa_{h_1}^2}{\kappa_{h_1}^2}\right\} - \left\{\frac{m_3^2}{\kappa_{h_3}^2} + \frac{\kappa_{h_3}^2}{\kappa_{h_3}^2}\right\} = \frac{1}{\omega_1^2} - \frac{1}{\omega_3^2}. \tag{103}$$

This was, in fact, one of the conditions for zero vertical vorticity production at $O(\epsilon)$ deduced in the previous section. This implies that,

$$\frac{\partial b_2}{\partial t_1} \equiv 0, \tag{104}$$

and $b_2 \equiv B_2$. Therefore, the equations for a_1 and a_3 can be readily combined into a single equation for, say a_1,

$$\frac{\partial^2 a_1}{\partial t_1^2} + \Gamma'^2 b_2 b_2^* a_1 = 0. \tag{105}$$

We incorporate $b_2 b_2^* = B_2^2$ into the coefficient

$$\Gamma^2 \equiv \Gamma'^2 B_2^2. \tag{106}$$

Since Γ^2 is always real and positive, the solutions are purely oscillatory. Thus the $O(1)$ solutions are,

$$w_0 = \cos\Gamma t_1 \sin\theta_1 - \sin\Gamma t_1 \cos\theta_3, \tag{107}$$

for the wave part and,
$$\Psi_0 = B_2 \sin\theta_2, \tag{108}$$

for the vortical mode.

Let us now examine the expression for the interaction coefficient Γ. In spherical coordinates, (97) is written,

$$\Gamma = \tfrac{1}{2}(B_2 \kappa_1 \kappa_3 \sin^2\gamma \sin(\phi_3 - \phi_1))\{\cos^2\gamma\cos(\phi_3 - \phi_1) + \sin^2\gamma\}, \tag{109}$$

where κ_1 and κ_3 are the respective magnitudes of the two wavenumbers, γ is the angle to the vertical of the two waves (since they have equal frequencies, the angle that each makes to the vertical is equal), and ϕ_1 and ϕ_3 are the respective azimuthal angles of waves 1 and 3.

$$k_i = \kappa_i \sin\gamma\cos\phi_i, \tag{110}$$

$$l_i = \kappa_i \sin\gamma\sin\phi_i, \tag{111}$$

$$m_i = \kappa_i \cos\gamma, \tag{112}$$

where $i = 1$ and 3. Several limiting cases are identified:

Case (i): $\phi_3 - \phi_1 = 0$

This corresponds to the two waves lying in the same vertical plane. The two waves coexist but do not exchange any energy.

Case (ii): $\phi_3 - \phi_1 = \tfrac{1}{2}\pi$

The two wavenumbers have orthogonal horizontal projections. Γ attains a maximum value for fixed β_2 and simplifies to $2\kappa_1 \kappa_2 \sin^4\gamma$.

The vortical mode here acts as a catalyst in that its presence is necessary to set up the interaction, yet it does not actually participate in the energy exchange. This type of behaviour is reminiscent of the resonant interaction between a slowly varying horizontal shear and two internal waves, first investigated by Phillips (1968). In Phillips' triad, the horizontal shear interacts with two internal waves of equal frequencies but with opposite inclination to the vertical. The wavenumbers form the two equal sides of an isosceles triangle. As in the present case, the shear merely acts as a catalyst. Our triad is of a more general nature, however, since no restrictions exist on the vertical direction of propagation of the two waves. There are many vortical modes κ_2 which will interact with a given internal wave κ_1 to excite a wave κ_3 of equal frequency. Another difference is that the present triad is not restricted to lying in a vertical plane. In fact, as was shown, the interaction coefficients vanish if the wavenumbers happen to lie in a vertical plane. This resonance is therefore inherently three-dimensional and different from Phillips' triad.

The analysis as it stands does not provide any clues as to whether preferred interactions exist. Nor can we, at this point, draw any conclusions about the relative importance of this class of resonances as compared with induced diffusion, elastic scattering and parametric subharmonic instability resonances (McComas & Bretherton 1977). It is nonetheless conceivable that the predicted internal wave spectra for both atmosphere and ocean could be significantly modified if this new class of resonances is taken into account, for it provides an additional mechanism through which wave energy can redistribute itself in spectral space. One might expect that it would tend to isotropize the spectrum. Furthermore, these results appear to substantiate the conclusions drawn by Métais & Herring (1989) regarding the strength of wave/vortical mode interactions in their numerical simulations of forced stably-stratified turbulence at small Froude numbers. They found that, at small Froude numbers, waves and vortical modes exchange very little energy. Since no net energy is exchanged in the wave/vortex resonance, their results are consistent with the present findings.

5. The vortical mode/vortical mode interaction

We now examine the interaction of two vortical modes. Velocity scales and lengthscales are based, as in the previous examples, on the initial conditions. The pressure is scaled as a dynamic pressure, $[P] = \rho_b U^2$, and the buoyancy as ϵNU. The only modification that this scaling brings to the governing equations is that the nonlinear terms involving buoyancy now appear at $O(\epsilon^2)$ rather than at $O(\epsilon)$.

Consider the following conditions,

$$\nabla_h^2 \Psi(\boldsymbol{x},0) = -B_1 \kappa_{h_1}^2 \sin(\boldsymbol{\kappa}_1\cdot\boldsymbol{x}) - B_2 \kappa_{h_2}^2 \sin(\boldsymbol{\kappa}_2\cdot\boldsymbol{x}), \tag{113}$$

$$w(\boldsymbol{x},0,0) = 0, \tag{114}$$

$$\frac{\partial w}{\partial t} = -\epsilon \frac{\partial p}{\partial z} + \epsilon b(\boldsymbol{x},0,0). \tag{115}$$

Again, we assume the simplest solutions at the lowest order,

$$\nabla_h^2 \Psi_0(\boldsymbol{x},t_0,t_1) = -B_1(t_1)\,\kappa_{h_1}^2 \sin\theta_1 - \beta_2(t_1)\,\kappa_{h_2}^2 \sin\theta_2, \tag{116}$$

with, $\beta_1(t=0) = B_1$, $\beta_2(t=0) = B_2$ and $\theta_i = \boldsymbol{\kappa}_i\cdot\boldsymbol{x} + \mu_i(t_1)$. For the wave field, we have

$$w_0(\boldsymbol{x},t_0,t_1) = 0. \tag{117}$$

At $O(\epsilon)$, the initial conditions are,

$$\nabla_h^2 \Psi_1(\boldsymbol{x},0,0) = 0, \tag{118}$$

$$w_1(\boldsymbol{x},0,0) = 0, \tag{119}$$

$$\frac{\partial w_1}{\partial t_0} = -\frac{\partial p_0}{\partial z} + b_0. \tag{120}$$

Since Ψ_0 is independent of the fast time t_0, the vortex/vortex term in the $O(\epsilon)$ wave equation remains zero. Thus at $O(\epsilon)$, no forcing of the wave field occurs as a result of the vortex/vortex interaction. If pressure and buoyancy are in perfect hydrostatic equilibrium, then the right-hand side of (120) will vanish. Initial hydrostatic balance also implies that the horizontal pressure gradient is equal to the nonlinear term, as can be seen from the equation for the pressure,

$$\nabla^2 p_0 = \frac{\partial b_0}{\partial z} - 1 \cdot (\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0), \tag{121}$$

which reduces to

$$\nabla_h p_0 = -(\boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0), \tag{122}$$

when the vertical pressure gradient and the buoyancy are equal. In this case, the initial conditions satisfy the steady-state equations and no adjustment takes place. Thus, no waves result.

If the initial vertical acceleration is non-zero, on the other hand, we have

$$w_1 = \iiint_{-\infty}^{\infty} [\alpha_\kappa(\kappa, t_1)\exp(\mathrm{i}\omega t_0) + \gamma_\kappa(\kappa, t_1)\exp(-\mathrm{i}\omega t_0)]\exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\kappa}. \tag{123}$$

The behaviour of the constants α_κ and γ_κ at $t = 0$ is determined from the initial conditions,

$$\alpha_\kappa(\boldsymbol{\kappa}, 0) = \frac{1}{2\mathrm{i}\omega}\left(\frac{\partial p_0}{\partial z} - b_0\right)\exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x}), \tag{124}$$

$$\gamma_\kappa(\boldsymbol{\kappa}, 0) = -\alpha_\kappa(\boldsymbol{\kappa}, 0). \tag{125}$$

The amplitude of the resulting wave field is proportional to the deviation from hydrostatic balance of the initial flow. Furthermore, it scales as the Froude number.

We now proceed to eliminate secular terms from the vorticity equation by setting

$$\frac{\partial}{\partial t_1}\nabla_h^2 \Psi_0 + \frac{\partial(\Psi_0, \nabla_h^2 \Psi_0)}{\partial(x,y)} = 0. \tag{126}$$

However, given the simple form of Ψ_0, the only possible solutions to (126) are the trivial solutions $\beta_1 = \beta_2$ and $\beta_1 = B_1$, $\beta_2 = B_2$. Furthermore, unlike the examples of §§3 and 4, adding contributions to Ψ_0 from the $\boldsymbol{\kappa}_1 \pm \boldsymbol{\kappa}_2$ modes will not suffice. Energy will quickly spread to higher harmonics. This is because, in terms of perturbation analysis, the vortex/vortex interaction is inherently resonant and broadbanded. The resonance condition on the frequencies is automatically satisfied since all vortical modes have zero frequency. Therefore, the only condition needed in order for three vortical modes to form a resonant triad is that their wave vectors form a triangle ($\boldsymbol{\kappa}_1 \pm \boldsymbol{\kappa}_2 = \boldsymbol{\kappa}_3$). The interaction of any two vortical modes will resonantly excite two additional modes which, in turn, will interact with others and so on. The reason for this behaviour is that, as Ψ_0 satisfies (126), it is fully nonlinear on its own timescale (T_1), and the perturbation method is not appropriate.

We shall not carry this example any further. We have shown that the vortex/vortex interaction is very broadbanded. Energy initially concentrated in two modes will rapidly spread out over the whole Fourier spectrum. If the flow is initially out of hydrostatic balance, an adjustment will take place and some energy will be released as waves, whose amplitudes will scale as the Froude number and as the deviation from the hydrostatic state. Any waves generated by the interaction of two vortical modes *per se* will appear at $O(\epsilon^2)$ and thus be negligible.

6. Conclusion

We have re-examined the problem of weakly nonlinear interactions in strongly-stratified blows, taking into account the presence of both vortical modes and internal waves. Taking advantage of the two inherent timescales that govern the evolution of the flow, a rigorous, multiple-scale mathematical model has been formulated.

We have sought to clarify the important distinction between vertical and potential vorticities in the nonlinear regime. Only in the linear approximation are the two synonymous, and it is the potential vorticity rather than the vertical vorticity which must be used in distinguishing vortical modes from internal waves. Whereas a weakly nonlinear wave/wave interaction may result in the production of some vertical vorticity, it cannot alter the potential vorticity of the flow.

In addition to wave/wave interactions, wave/vortex and vortex/vortex interactions also exist and they are likely to affect the dynamics of strongly stratified flows. Like their wave/wave counterpart, the effects of these two additional interactions are most strongly felt when resonance conditions are met. We have identified resonant triads for both wave/vortex and vortex/vortex interactions.

The wave/vortex resonant triad involves two equal-frequency waves and one vortical mode. The presence of the vortical mode is crucial to the set-up of this resonance, yet it does not engage in any energy exchange with the wave field, acting instead as a catalyst in moving energy from one wave to the other. This transfer of energy is periodic in time, with the frequency directly proportional to the vortical mode amplitude, the magnitudes of the wave wavenumbers and the frequency of the two waves.

In the vortex/vortex interaction, the resonance conditions are automatically satisfied by any three wave vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ and $\boldsymbol{\kappa}_3$ if $\boldsymbol{\kappa}_3 = \boldsymbol{\kappa}_1 \pm \boldsymbol{\kappa}_2$. The resonance condition on the frequencies is identically satisfied since the frequency of any vortical mode, regardless of its wavenumber, is zero. Consequently, the vortex/vortex interaction is very broadbanded, the energy spreading out in spectral space through a cascade of resonant interactions as described above. On its own slow timescale, the evolution of the vortical field is inherently nonlinear. If the wave vectors of two interacting vortical modes happen to lie in the same vertical plane or have equal horizontal components, the resonance is suppressed and vortical energy remains localized in the two modes. As long as the two interacting vortical modes exhibit vertical variability, their interaction (resonant or non-resonant), through the set-up of vertical pressure gradients, will result in the generation of some small-amplitude internal waves with corresponding wave vectors $\boldsymbol{\kappa}_1 \pm \boldsymbol{\kappa}_2$. Strong interactions subsequently take place between these waves and the vortical field when wave/vortex resonance conditions are encountered. Energy eventually gets redistributed throughout the entire spectrum of waves as a result of these resonances.

In the context of geophysical fluid applications, the wave/vortex and vortex/vortex resonances may be significant. The former provides an additional mechanism capable of redistributing wave energy in spectral space, while the latter offers a source of internal wave generation. Because of its somewhat elusive nature, the wave/vortex resonance has until now escaped notice. Since it does not result in a net exchange of energy between wave and vortical fields, it has not been detected in direct numerical simulations involving broadbanded flows. Further studies are needed at this point to establish how the presence of the wave/vortex and vortex/vortex interactions affect the behaviour of existing weakly nonlinear theories. In particular, calculations of internal wave spectra should be re-examined with these added features.

Furthermore, one can envisage many possible situations in which wave and vortical modes might interact in a strongly nonlinear fashion that cannot be handled with our inviscid perturbation scheme. As demonstrated in §5, the vortex/vortex interaction does not lend itself very well to a weakly nonlinear model.

We conclude by stating that the results of this study provide further evidence that the role of the vortical mode in influencing the evolution of strongly stratified flows may be significant and should not be neglected. Further work is needed at this point to establish the relative importance of the wave/vortex and vortex/vortex resonances compared to the wave/wave resonances.

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